The Hausdorff dimension of average conformal repellers under random perturbation

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Abstract. We prove that the Hausdorff dimension of an average conformal repeller is stable under random perturbations. Our perturbation model uses the notion of a bundle random dynamical system.

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1 Introduction.

In the dimension theory of dynamical system, only the Hausdorff dimension of invariant sets of conformal dynamical system is well understood. Since the work of Bowen, who

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was the first to express the Hausdorff dimension of an invariant set as a solution of an equation involving topological pressure. Ruelle [18] refined Bowen's method and get the following result. If J is a mixing repeller for a $C^{1+\alpha}$ conformal expanding map f on a Riemannian manifold M, then the Hausdorff dimension of J can be obtained as the zero t_0 of $t \mapsto \pi_f(-t \log ||D_x f||)$, where π_f denotes the topological pressure functional. This statement is known as the Bowen-Ruelle formula, and we sometimes call the equation involving topological pressure Bowen equation. And Gatzouras and Peres relaxed the smoothness $C^{1+\alpha}$ to C^1 in [12].

Recently, different version of topological pressure has become an useful tool in calculating the Hausdorff dimension of a non-conformal repeller. For C^1 non-conformal repellers, Zhang used singular values of the derivative $D_x f^n$ for all $n \in \mathbb{N}$, to define a new equation which involves the limit of a sequence of topological pressure, then he showed that the upper bound of the Hausdorff dimension of repeller was given by the unique solution of the equation, see [20] for details. Barreira considered the same problem in [2]. By using the non-additive thermodynamic formalism which was introduced in [3] and singular value of the derivative $D_x f^n$ for all $n \in \mathbb{N}$, he gave an upper bound of box dimension of repeller under the additional assumptions that the map was $C^{1+\alpha}$ and α-bunched. This automatically implies that for Hausdorff dimension. In [7], by using the sub-additive topological pressure which was studied in [8], the author proved that the upper bound of Hausdorff dimension for C^1 non-conformal repellers obtained in [2, 11, 20] were same and it was the unique root of Bowen equation for sub-additive topological pressure, we point out that the map is only need to be C^1 without any additional condition in [7].

In [1], the authors introduced the notion of average conformal repeller in the deterministic dynamic systems which was a generalization of quasi-conformal and asymptotically conformal repeller in [3, 17], and they proved that the Hausdorff dimension and box dimension of average conformal repellers was the unique root of Bowen equation for sub-additive topological pressure. In that paper, the map is only needed C^1, without any additional condition.

For random repellers, Kifer proved that the Hausdorff dimension of a measurable random conformal repeller was the root of the Bowen equation which can be seen as a random version of the deterministic case, see [13] for details. And in [4], the authors generalized this result to almost-conformal case. In [21], using the idea in the deterministic case [1], authors introduced the notion of random average conformal repeller, and they proved that the Hausdorff dimension of random average conformal repellers was the unique root of Bowen equation for random sub-additive topological pressure which was studied in [22].

Motivated by the work in [4], where the authors showed that the Hausdorff dimension of the conformal repeller was stable under random perturbation, we consider

a random perturbation of the deterministic average conformal repeller which is modeled using the notion of a bundle random dynamical system(RDS for short). Namely, let ϑ be an ergodic invertible transformation of a Lebesgue space $(\Omega, \mathcal{W}, \mathbb{P})$ and consider a measurable family $T = \{T(\omega) : M \to M\}$ of $C^{1+\alpha}$ maps, that is to say, $(\omega, x) \mapsto T(\omega)x$ is assumed to be measurable. This determines a differentiable RDS via $T(n, \omega) := T(\vartheta^{n-1}\omega) \circ \cdots \circ T(\vartheta\omega) \circ T(\omega)(n \in \mathbb{N})$. Further, Let $E \subset \Omega \times M$ be a measurable set such that all ω-sections $E_\omega := \{x \in M : (\omega, x) \in E\}$ are compact. If $\mathcal{K}$ denotes the collection of all compact subsets of M endowed with the Hausdorff topology, this is equivalent to saying that $\mathcal{K}$-valued multifunction $\omega \mapsto E_\omega$ is measurable. Here and in what follows we think of E_ω being equipped with the trace topology, i.e. an open set $A \subset E_\omega$ is of the form $A = B \cap E_\omega$ with some open set $B \subset M$. We call E is T-invariant if $T(\omega)E_\omega = E_{\vartheta\omega}$ $\mathbb{P} - a.s.$, and in this situation the Hausdorff dimension of the fiber E_ω is a $\mathbb{P}$-a.s. constant, see [10]. The map $\Theta : E \to E$ is defined by $\Theta(\omega, x) = (\vartheta\omega, T(\omega)x)$, and we call it the skew product transformation.

The aim of this paper is to make rigorous the statement that if a bundle RDS is close to an average conformal expanding map on a repeller then the corresponding Hausdorff dimension are close.

The paper is organized as follows. In section 2, we will recall the main result in [1]. In section 3, we introduce some random notions and our model of random perturbation, we point out that this was essentially inspired by a remarkable result of Liu [15]. In section 4, we formulate and prove our main result which says that the Hausdorff dimension of an average conformal repeller is stochastically stable.

2 Dimension of average conformal repeller

In this section, we will recall the notion of sub-additive topological pressure and the main result in [1] which says that the Hausdorff dimension of an average conformal repeller can be given by the unique root of the sub-additive topological pressure. Moreover, we will give some preliminary results.

Let $f : X \to X$ be a continuous map on a compact space X with metric d. A subset $E \subset X$ is called an (n, ϵ)-separated set with respect to f if $x \neq y \in E$ implies $d_n(x, y) := \max_{0 \le i \le n-1} d(f^i x, f^i y) > \epsilon$. Let $\mathcal{F} = \{\phi_n\}_{n \ge 1}$ denote a sub-additive potential on X, that is to say $\phi_n : X \to \mathbb{R}$ is continuous for each $n \in \mathbb{N}$ and satisfying

$$\phi_{n+m}(x) \le \phi_n(x) + \phi_m(f^n(x)),\ \forall n, m \in \mathbb{N}, x \in X.$$

Following the way in [8], we define the sub-additive topological pressure

$$\pi_f(\mathcal{F}, n, \epsilon) = \sup \left\{ \sum_{x \in E} \exp \phi_n(x) : E\ is\ an\ (n, \epsilon) - separated\ subset\ of\ X \right\}$$

and then call

$$\pi_f(\mathcal{F}) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log \pi_f(\mathcal{F}, n, \epsilon)$$

the sub-additive topological pressure of $\mathcal{F}$ with respect to f. If there is no confusion caused, we simply call $\pi_f(\mathcal{F})$ the sub-additive topological pressure of $\mathcal{F}$.

Remark 1. *(1)When the continuous potential $\mathcal{F} = \{\phi_n\}$ on X is additive, i.e. $\phi_n(x) = \sum_{i=0}^{n-1}\phi(f^i x)$ for some continuous function $\phi : X \to \mathbb{R}$, then $\pi_f(\mathcal{F})$ is the classical topological pressure, see [19] for details, and we denote it simply by $\pi_f(\phi)$; (2)When the continuous potential $\mathcal{F} = \{\phi_n\}$ on X is sup-additive, that is to say, $\phi_{n+m}(x) \geq \phi_n(x) + \phi_m(f^n x)$, $\forall n, m \in \mathbb{N}, x \in X$, we also can define the sup-additive topological pressure. And the pressures are equal under some special case, see [1].*

Let $\mathcal{M}(X, f)$ denote the space of all f-invariant Borel probability measures and $\mathcal{E}(X, f)$ denote the subset of $\mathcal{M}(X, f)$ with ergodic measures. For $\mu \in \mathcal{M}(X, f)$, let $h_\mu(f)$ denote the measure-theoretic entropy of f with respect to μ, and let $\mathcal{F}_*(\mu)$ denote the following limit

$$\mathcal{F}_*(\mu) = \lim_{n\to\infty}\frac{1}{n}\int \phi_n \mathrm{d}\mu.$$

The relation between $\pi_f(\mathcal{F}), h_\mu(f)$ and $\mathcal{F}_*(\mu)$ is given by the following variational principle which is proved in [8], and the random version of the following theorem is proved in [22].

Theorem 2.1 (Variational principle)**.** *Let $\mathcal{F}$ be a sub-additive potentials on a compact metric space X, and $f : X \to X$ is a continuous transformation, then*

$$\pi_f(\mathcal{F}) = \begin{cases} -\infty, & if\ \mathcal{F}_*(\mu) = -\infty\ for\ all\ \mu \in \mathcal{M}(X, f) \\ \sup\{h_\mu(f) + \mathcal{F}_*(\mu) : \mu \in \mathcal{M}(X, f), \mathcal{F}_*(\mu) \neq -\infty\}, & otherwise. \end{cases}$$

Proposition 2.1. *let $f_i : X_i \to X_i (i = 1, 2)$ be a continuous map of a compact metric space (X_i, d_i), and $\mathcal{F} = \{\phi_n\}$ is a sub-additive potential on X_2. If $\varphi : X_1 \to X_2$ is a surjective continuous map with $\varphi \circ f_1 = f_2 \circ \varphi$ then $\pi_{f_2}(\mathcal{F}) \leq \pi_{f_1}(\mathcal{F} \circ \varphi)$, and if φ is a homeomorphism then $\pi_{f_2}(\mathcal{F}) = \pi_{f_1}(\mathcal{F} \circ \varphi)$, where $\mathcal{F} \circ \varphi = \{\phi_n \circ \varphi\}$.*

Proof. We first check that $\mathcal{F} \circ \varphi$ is indeed a sub-additive potential on the compact metric space X_1. In fact

$$\phi_{n+m} \circ \varphi(x) \leq \phi_n(\varphi x) + \phi_m(f_2^n \varphi x) = \phi_n \circ \varphi(x) + \phi_m \circ \varphi(f_1^n x)$$

the equality follows from the fact that $\varphi \circ f_1 = f_2 \circ \varphi$.

Let $\epsilon > 0$ and choose $\delta > 0$ such that $d_2(\varphi(x), \varphi(y)) > \epsilon$ implies $d_1(x, y) > \delta$, this fact follows from the uniform continuity of φ.

Let E be an (n, ϵ)-separated set with respect to f_2. Since φ is surjective, there exists a subset $F \subset X_1$ so that $\varphi|_F : F \to E$ is a bijection. It follows from the above observation that F is an (n, δ)-separated set with respect to f_1. Hence, we have

$$\begin{aligned}
\pi_{f_2}(\mathcal{F}, n, \epsilon) \quad &= \sup\{\textstyle\sum_{x\in E} \exp \phi_n(x) : E \text{ is an } (n, \epsilon)\text{-separated subset of } X_2\} \\
&= \sup\{\textstyle\sum_{y\in F} \exp \phi_n(\varphi y) : E \text{ is an } (n, \epsilon)\text{-separated subset of } X_2 \\
&\qquad \text{and } \varphi|_F : F \to E \text{ is a bijection}\} \\
&\leq \sup\{\textstyle\sum_{y\in F} \exp \phi_n(\varphi y) : F \text{ is an } (n, \delta)\text{-separated subset of } X_1\} \\
&= \pi_{f_1}(\mathcal{F} \circ \varphi, n, \delta)
\end{aligned}$$

Since $\epsilon \to 0$ then $\delta \to 0$, then we can have

$$\pi_{f_2}(\mathcal{F}) \leq \pi_{f_1}(\mathcal{F} \circ \varphi)$$

If φ is a homeomorphism then we can apply the above with $f_1, f_2, \varphi, \mathcal{F}$ replaced by $f_2, f_1, \varphi^{-1}, \mathcal{F} \circ \varphi$ respectively to give $\pi_{f_2}(\mathcal{F}) \geq \pi_{f_1}(\mathcal{F} \circ \varphi)$. Thus the proof is finished. $\square$

Proposition 2.2. *Let $f : X \to X$ be a continuous map on a compact metric space, and $\phi : X \to \mathbb{R}$ is a continuous function on X. Suppose $\varphi_\epsilon : X \to \mathbb{R}$ is a continuous function on X for every $\epsilon > 0$ and $\lim_{\epsilon\to 0} \varphi_\epsilon = \phi$, then*

$$\lim_{\epsilon\to 0} \pi_f(\varphi_\epsilon) = \pi_f(\phi).$$

Proof. This immediately follows from the continuity of the classical topological pressure. $\square$

Now we introduce the definition of average conformal repeller. And the dimension of the repeller can be obtained by the unique root of the corresponding sub-additive topological pressure.

Let M be a C^∞ m-dimensional Riemannian manifold. Let U be an open subset of M and $f : U \to M$ be a C^1 map. Suppose $J \subset U$ is a compact f-invariant subset. Let $\mathcal{M}(f|_J)$, $\mathcal{E}(f|_J)$ denote the set of all f-invariant measures and the set of all ergodic invariant measures supported on J respectively. For any $\mu \in \mathcal{E}(f|_J)$, by the Oseledec multiplicative ergodic theorem (see [16]), we can define Lyapunov exponents $\lambda_1(\mu) \leq \lambda_2(\mu) \leq \cdots \leq \lambda_m(\mu)$, $m = \dim M$.

Definition 2.1. *A compact f-invariant set J is called an average conformal repeller for f if for any $\mu \in \mathcal{E}(f|_J)$, $\lambda_1(\mu) = \lambda_2(\mu) = \cdots = \lambda_m(\mu) > 0$.*

Remark 2. *We point out that if J is an average conformal repeller for f, it is indeed a repeller in the usual way(see [9]) that: $\exists \lambda > 1, C > 0$ such that for all $x \in J$ and $v \in T_xM$*

$$||D_xf^n(v)|| \geq C\lambda^n||v||, \quad \forall n \geq 1.$$

Proposition 2.3. *If J is an average conformal repeller for f, then*

$$\pi_f(\Phi) = \lim_{n\to\infty} \frac{1}{n} \pi_{f^n}(-\log \|D_x f^n\|)$$

where $\Phi = \{-\log \|D_x f^n\|\}_{n\geq 1}$ is a sup-additive potential and $\pi_f(\Phi)$, $\pi_{f^n}(-\log \|D_x f^n\|)$ denote the sup-additive topological pressure of Φ with respect to f, classical topological pressure of $-\log \|D_x f^n\|$ with respect to f^n respectively.

Proof. Let $\Psi = \{-\log m(D_x f^n)\}_{n\geq 1}$ denotes the sub-additive potential. First note that by the definition of topological pressure, we have

$$\frac{1}{k}\pi_{f^k}(-\log \|D_x f^k\|) \leq \frac{1}{k}\pi_{f^k}(-\log m(D_x f^k)), \ \forall k \geq 1.$$

And since J is an average conformal repeller, the measure-theoretic entropy map $\mu \mapsto h_\mu(f)$ is upper-semi-continuous by remark 2. By proposition 2.2 in [7], we have

$$\lim_{k\to\infty} \frac{1}{k}\pi_{f^k}(-\log m(D_x f^k)) = \pi_f(\Psi)$$

where $\pi_f(\Psi)$ denotes the sub-additive topological pressure of Ψ with respect to f. Thus we have

$$\limsup_{k\to\infty} \frac{1}{k}\pi_{f^k}(-\log \|D_x f^k\|) \leq \pi_f(\Psi) = \pi_f(\Phi), \tag{2.1}$$

where the last equality is proved in [1] since J is an average conformal repeller for f.

On the other hand, for any $\mu \in \mathcal{M}(f|_J) \subset \mathcal{M}(f^k|_J)$, we have

$$\begin{aligned} h_\mu(f) + \Phi_*(\mu) &= \lim_{k\to\infty} \frac{1}{k}(h_\mu(f^k) + \int -\log \|D_x f^k\| \mathrm{d}\mu) \\ &\leq \liminf_{k\to\infty} \frac{1}{k}\pi_{f^k}(-\log \|D_x f^k\|), \end{aligned}$$

the last inequality is follows from the classical variational principle for additive topological pressure of $-\log \|D_x f^k\|$ with respect to f^k, see [19]. Again because J is an average conformal repeller, by the variational principle for the sup-additive topological pressure(see [1]), we have

$$\pi_f(\Phi) \leq \liminf_{k\to\infty} \frac{1}{k}\pi_{f^k}(-\log \|D_x f^k\|). \tag{2.2}$$

Thus the desired result immediately follows from (2.1) and (2.2). □

The dimension of an average conformal repeller can be given by the following theorem in [1].

Theorem 2.2. *Let f be C^1 dynamical system and J be an average conformal repeller for f, then the Hausdorff dimension of J is zero of $t \mapsto \pi_f(-t\Psi)$, where $\Psi = \{\log m(D_x f^n) : x \in J, n \in \mathbb{N}\}$ and $m(A) = ||A^{-1}||^{-1}$.*

3 Random notations

In this section, we will give some random notions and some well-known results. Firstly, let $(\Omega, \mathcal{W}, \mathbb{P})$ and ϑ, E, T be described in section 1, and let $\mathcal{M}^1_{\mathbb{P}}(E,T)$ denote the space of Θ-invariant measures with marginal $\mathbb{P}$ on Ω of the RDS, $\mathcal{E}^1_{\mathbb{P}}(E,T)$ denote the subset of $\mathcal{M}^1_{\mathbb{P}}(E,T)$ with ergodic measures of the RDS.

Let $L^1_E(\Omega, C(M))$ denote the collection of all integrable random continuous functions on fibers, i.e. a measurable $f : E \to \mathbb{R}$ is a member of $L^1_E(\Omega, C(M))$ if $f(\omega) : E_\omega \to \mathbb{R}$ is continuous and $\|f\|_1 := \int \|f(\omega)\| \mathrm{d}\mathbb{P}(\omega) < \infty$, where $\|f(\omega)\| = \sup_{x \in E_\omega} |f(\omega, x)|$. If we identify f and g provided $\|f - g\|_1 = 0$, then $L^1_E(\Omega, C(M))$ becomes a Banach space with the norm $\|\cdot\|_1$. A family $\Phi = \{\varphi_n\}_{n \geq 1}$ of integrable random continuous functions on E is called sub-additive if for $\mathbb{P}$-almost all ω,

$$\varphi_{n+m}(\omega, x) \leq \varphi_n(\omega, x) + \varphi_m(\Theta^n(\omega, x)) \text{ for all } n, m \in \mathbb{N}, x \in E_\omega.$$

Let $\epsilon : \Omega \to (0, 1]$ be a measurable function. A set $F \subset E_\omega$ is said to be (ω, ϵ, n)-separated for T, if $x, y \in F, x \neq y$ implies $y \notin B_\omega(n, x, \epsilon)$, where $B_\omega(n, x, \epsilon) := \{y \in E_\omega : d(T(k, \omega)x, T(k, \omega)y) < \epsilon(\vartheta^k \omega) \ for\ 0 \leq k \leq n-1\}$ and d is the given metric on M.

Let $\Phi = \{\varphi_n\}_{n \geq 1}$ be a sub-additive function sequence with $\varphi_n \in L^1_E(\Omega, C(M))$ for each n. As usual, we put

$$\begin{aligned}
&\pi_T(\Phi)(\omega, \epsilon, n) = \sup\{\sum_{x \in F} e^{\varphi_n(\omega, x)} : F \text{ is an } (\omega, \epsilon, n)\text{-separated subset of } E_\omega\} \\
&\pi_T(\Phi)(\epsilon) = \limsup_{n \to \infty} \tfrac{1}{n} \int \log \pi_T(\Phi)(\omega, \epsilon, n) \mathrm{d}\mathbb{P}(\omega) \\
&\pi_T(\Phi) = \lim_{\epsilon \downarrow 0} \pi_T(\Phi)(\epsilon)
\end{aligned}$$

The last quantity is called the sub-additive topological pressure of Φ with respect to T. We just mention that the above definition is reasonable, see [22] for details.

Remark 3. *(i) If the function sequence $\Phi = \{\varphi_n\}$ can be written as $\varphi_n(\omega, x) = \sum_{i=0}^{n-1} \varphi(\Theta^i(\omega, x))$ for some function $\varphi \in L^1_E(\Omega, C(M))$, then we call $\pi_T(\Phi)$ the random additive topological pressure, see [4, 14] for details, denote it simply by $\pi_T(\varphi)$. (ii) Since $\mathbb{P}$ is ergodic in the model which we consider, so the limits in the above definition will not change $\mathbb{P}$-almost everywhere without integrating against $\mathbb{P}$.*

Lemma 3.1. *For $i = 1, 2$, let X_i be compact metric spaces, E_i measurable bundles over Ω with compact fibers in X_i, and φ_i topological bundle random dynamical systems on E_i. If $\psi = \{\psi(\omega) : E^1_\omega \to E^2_\omega\}$ is a family of homeomorphism between E^1_ω and E^2_ω satisfying $\varphi_2(\omega) \circ \psi(\omega) = \psi(\vartheta\omega) \circ \varphi_1(\omega)$, $\mathbb{P} - a.s.$, and $\mathcal{F} = \{f_n\}_{n \geq 1}$ is a sub-additive potential in $L^1_{E_2}(\Omega, C(X_2))$ then*

$$\pi_{\varphi_2}(\mathcal{F}) = \pi_{\varphi_1}(\mathcal{F} \circ \psi)$$

where $\mathcal{F}\circ\psi = \{f_n\circ\psi\}_{n\geq 1}$ *denotes the member of* $L^1_{E_1}(\Omega, C(X_1))$ *defined by* $f_n(\omega, \psi(\omega)x)$ *for each* $n \geq 1$.

Proof. We first check the new defined potential $\mathcal{F}\circ\psi = \{f_n\circ\psi\}_{n\geq 1}$ is indeed sub-additive. Precisely, we have

$$\begin{aligned} f_{n+m}\circ\psi(\omega,x) &= f_{n+m}(\omega,\psi(\omega)x) \\ &\leq f_n(\omega,\psi(\omega)x) + f_m(\vartheta^n\omega, \varphi_2(n,\omega)\psi(\omega)x) \\ &= f_n\circ\psi(\omega,x) + f_m(\vartheta^n\omega, \psi(\vartheta^n\omega)\varphi_1(n,\omega)x) \\ &= f_n\circ\psi(\omega,x) + f_m\circ\psi(\vartheta^n\omega, \varphi_1(n,\omega)x). \end{aligned}$$

Let $\mu \in \mathcal{M}^1_{\mathbb{P}}(E_1,\varphi_1)$ and write μ_ψ for the member of $\mathcal{M}^1_{\mathbb{P}}(E_2,\varphi_2)$ defined by $\psi(\omega)_*\mu_\omega$. We have $h^{(r)}_\mu(\varphi_1) = h^{(r)}_{\mu_\psi}(\varphi_2)$ (see theorem 2.2.2 in [5]) and $\lim_{n\to\infty}\frac{1}{n}\int f_n\circ\psi \mathrm{d}\mu = \lim_{n\to\infty}\frac{1}{n}\int f_n \mathrm{d}\mu_\psi$, by the variational principle of random sub-additive topological pressure in [22] we have $\pi_{\varphi_1}(\mathcal{F}\circ\psi) \leq \pi_{\varphi_2}(\mathcal{F})$. By symmetry we get the reverse inequality and hence the desired result. □

Definition 3.1. *Let* T *be a RDS over* ϑ*. A generator of RDS* T *is a family* $\mathcal{A} = \{\mathcal{A}(\omega) = (A_i(\omega)) : \mathcal{A}(\omega)$ *is an open cover of* $E_\omega\}$ *with*

(i) $\mathcal{A}(\omega)$ *is finite for all* $\omega\in\Omega$*;*

(ii) $\omega \mapsto d(x, A_i(\omega))$ *is measurable for all* $x\in M$ *and all* $i\in\mathbb{N}$*;*

(iiii) for each sequence $(A_n)_{n\in\mathbb{N}}$ *of sets we have* $A_n \in \mathcal{A}(\vartheta^n\omega)$ *for all* $n\in\mathbb{N}$ *implies that* $\bigcap_{n=0}^{\infty} T(n,\omega)^{-1}\bar{A}_n$ *contains at most one point.*

Definition 3.2. *Let* T *be a RDS over* ϑ*. We call* T *is (positive)expansive if there exists a* $(0,1)$*-valued random variable* Δ *such that*

$$d(T(n,\omega)x, T(n,\omega)y) \leq \Delta(\vartheta^n\omega) \ for\ all\ n\in\mathbb{N}$$

implies $x = y$.

Definition 3.3. *A generator* $\mathcal{A}$ *of a given RDS* T *is called a strong generator if*

$$\lim_{k\to\infty} \operatorname{diam} \bigvee_{i=0}^{k-1} T(i,\omega)^{-1}\mathcal{A}(\vartheta^i\omega) = 0 \ \ uniformly\ in\ \omega$$

An expansive RDS is said to be strongly expansive if it possesses a strong generator.

Let $U\subset M$ be an open subset of the Riemannian manifold M with $\bar{U}\in\mathcal{K}$ and let $C(U,M)$ denote the space of all continuous maps from U to M endowed with the compact open topology.

Definition 3.4. *Assume $f \in C(U, M)$ and $J \in \mathcal{K}$ with $fJ = J$. A family $\{T_\epsilon\}_{\epsilon>0}$ of $C(U, M)$-valued random variables is called a random perturbation of f on J if*

(i) $\lim_{\epsilon\to 0} T_\epsilon = f$ *in probability;*

(ii) there exists a family of $\mathcal{K}$-valued random variables $\{J_\epsilon\}_{\epsilon>0}$ such that

(a) for each $\epsilon > 0$ we have that $\mathbb{P}$-a.s. $T_\epsilon(\omega)J_\epsilon(\omega) = J_\epsilon(\vartheta\omega)$;

(b) $\lim_{\epsilon\to 0} J_\epsilon = J$ *in probability.*

$\{T_\epsilon\}_{\epsilon>0}$ is said to be structurally stable if there exists a family $\{h_\epsilon\}_{\epsilon>0}$ of $C(J, M)$-valued random variables such that

(iii) for each $\epsilon > 0$ we have that $h_\epsilon(\omega) : J \to J_\epsilon(\omega)$ is a homeomorphism and $T_\epsilon(\omega) \circ h_\epsilon(\omega) = h_\epsilon(\vartheta\omega) \circ f$ $\mathbb{P}$-a.s.;

(iv) $\lim_{\epsilon\to 0} h_\epsilon = id$ *in probability.*

See [4] for examples of strongly expansive bundle RDS and structurally stable random perturbation. Put $X = \overline{U}$ and let $C^\alpha(X, \mathbb{R})$ denote the space of all Hölder continuous functions on X with Hölder exponent α. We endow $C^\alpha(X, \mathbb{R})$ with the usual norm $|| \cdot ||_\alpha := || \cdot || + | \cdot |_\alpha$, where $|| \cdot ||$ is the sup-norm and $| \cdot |_\alpha$ is the least Hölder constant, namely, $|\varphi|_\alpha := \sup_{x,y\in X, x\neq y} \frac{|\varphi(x)-\varphi(y)|}{d(x,y)^\alpha}$. Then using proposition 2.2 we can get the following important proposition which is proved in [4], we cite here just for complete.

Proposition 3.1. *Let $\{T_\epsilon\}_{\epsilon>0}$ be a structurally stable random perturbation of f on J. And let $\{\varphi_\epsilon\}_{\epsilon>0}$ be a family of $C^\alpha(X, \mathbb{R})$-valued random variables satisfying*

$$\lim_{\epsilon\to 0} ||\varphi_\epsilon - \phi||_\alpha = 0 \text{ in } L^1(\mathbb{P})$$

for some $\phi \in C^\alpha(X, \mathbb{R})$. Then

$$\lim_{\epsilon\to 0} \pi_{T_\epsilon}(\varphi_\epsilon) = \pi_f(\phi).$$

4 Stability of the Hausdorff dimension under random perturbations

In this section we will prove that the Hausdorff dimension of the average conformal repeller is stable under suitable random perturbation.

The following proposition can be proved by slightly modification of the proof of theorem 1.1 in [15].

Proposition 4.1. *Assume J is an average conformal repeller for a $C^{1+\alpha}$ map $f : U \to M$. There exists a C^1 neighborhood $\mathcal{U}(f) \subset C^{1+\alpha}(U, M)$ of f such that the following holds:*
(i) For every random variable $T : \Omega \to \mathcal{U}(f)$ there exists a $\mathcal{K}$-valued random variable $J(\omega) \subset U$ satisfying $T(\omega)J(\omega) = J(\vartheta\omega)$, and a $C^0(U, M)$-valued random variable h such that each $h(\omega)$ is a homeomorphism between J and $J(\omega)$ and $T(\omega) \circ h(\omega) = h(\vartheta\omega) \circ f$ on J.
(ii) If $\{T_\epsilon : \Omega \to \mathcal{U}(f)\}_{\epsilon>0}$ is a family of random variables with $\lim_{\epsilon\to 0} T_\epsilon = f$ in probability(with respect to the C^1 distance), then $\lim_{\epsilon\to 0} h_\epsilon = \mathrm{id}$ in probability and thus $\lim_{\epsilon\to 0} J_\epsilon = J$ in probability. Here h_ϵ and J_ϵ are the corresponding objects associated to T_ϵ by (i).
In other words, each family $\{T_\epsilon\}_{\epsilon>0}$ of $\mathcal{U}(f)$-valued random variables with $\lim_{\epsilon\to 0} T_\epsilon = f$ in probability is a structurally stable random perturbation of f on J.

Now we state and prove our main result.

Theorem 4.1. *Assume J is an average conformal repeller for a $C^{1+\alpha}$ map $f : U \to M$. There exists a C^1 neighborhood $\mathcal{U}(f) \subset C^{1+\alpha}(U, M)$ of f such that the assertions of proposition 4.1 hold with the following additional property.*

If $\{T_\epsilon : \Omega \to \mathcal{U}(f)\}_{\epsilon>0}$ is a random perturbation of f with

$$\lim_{\epsilon\to 0} T_\epsilon = f \text{ in } L^1(\Omega, C^{1+\alpha}(U, M))$$

then

$$\lim_{\epsilon\to 0} \dim_H(J_\epsilon(\omega)) = \dim_H(J) \quad \mathbb{P}\text{-a.s.}$$

where $\dim_H(\cdot)$ denote the Hausdorff dimension of a set. Moreover, if $L \subset J$ is compact and f-invariant, then $\lim_{\epsilon\to 0} \dim_H(h_\epsilon(\omega)L) = \dim_H(L)$ $\mathbb{P}$-a.s.

Proof. In the following we will follow Bogenschütz and Ochs' proof [4] to obtain the desired result. Choose an open neighborhood V of J such that $\overline{V}$ is a compact subset of U. Then $\mathcal{U}(f)$ can be chosen in such a way that $J_\epsilon(\omega) \subset \overline{V}$ for every $\epsilon > 0, \omega \in \Omega$.

Fix $\epsilon > 0$. For $(\omega, x) \in \Omega \times \overline{V}$ set

$$\eta_\epsilon(\omega, x) := ||D_x T_\epsilon(\omega)|| \text{ and } \lambda_\epsilon(\omega, x) := m(D_x T_\epsilon(\omega)).$$

By taking appropriate $\mathcal{U}(f)$, we can assume that $\log \lambda_\epsilon, \log \eta_\epsilon \in L^1_{J_\epsilon}(\Omega, C(M))$.

For the clarity of the proof, we divide the proof into several steps.

Step 1: We claim that T_ϵ satisfies the following formula

$$\lambda_\epsilon(\omega, x) - K(\omega)d(x, y)^\alpha \le \frac{d(T_\epsilon(\omega)x, T_\epsilon(\omega)y)}{d(x, y)} \le \eta_\epsilon(\omega, x) + K(\omega)d(x, y)^\alpha \qquad (4.3)$$

for every $\omega \in \Omega$ and $x \neq y \in J_\epsilon(\omega)$, where $K : \Omega \to \mathbb{R}_+$ with $\log^+ K \in L^1(\mathbb{P})$. We will prove the inequality (4.3) in the rest of this step.

Choose $r_0 > 0$ such that $A := \{x : dist(x, \overline{V}) \leq r_0\} \subset U$. Define

$$K_0(\omega) = |DT_\epsilon(\omega)|_{\alpha,A} = \sup\left\{\frac{\|D_xT_\epsilon(\omega) - D_yT_\epsilon(\omega)\|}{d(x,y)^\alpha} : x, y \in A, x \neq y\right\}.$$

For the simplicity of notations, we restrict M to be the case of an open subset of $\mathbb{R}^d$, since the general case can be done via local coordinates. We let $|\cdot|$ denote the Euclidian norm on $\mathbb{R}^d$ and write T instead of $T_\epsilon(\omega)$ for convenience.

For $x, y \in J_\epsilon(\omega)$ with $0 < |x - y| < r_0$, we put $e := \frac{y-x}{|y-x|}$ and get that

$$\begin{aligned} |T(x) - T(y)| &= \left|\int_0^{|y-x|} D_{x+te}T(e)\ \mathrm{d}t\right| \leq \int_0^{|y-x|} \|D_{x+te}T\|\mathrm{d}t \\ &\leq |y-x| \sup\{\|D_{x+z}T\| : |z| \leq |y-x|\} \\ &\leq |y-x|(\|D_xT\| + K_0(\omega)|y-x|^\alpha). \end{aligned}$$

Thus, we get that

$$\frac{|T(x) - T(y)|}{|y-x|} \leq \eta_\epsilon(\omega, x) + K_0(\omega)|y-x|^\alpha.$$

On the other hand, we can get that

$$\begin{aligned} \frac{1}{|y-x|} & \left|\int_0^{|y-x|} D_{x+te}T(e)\ \mathrm{d}t\right| \\ & \geq \inf\{|Ae| : A \in \text{convex hull of } D_{x+te}T, 0 \leq t \leq |y-x|\} \\ & \geq |D_xT(e)| - \sup\{|Ae| : A \in \text{convex hull of } (D_{x+te}T - D_xT), \\ & \quad 0 \leq t \leq |y-x|\} \\ & \geq \lambda_\epsilon(\omega, x) - K_0(\omega)|y-x|^\alpha. \end{aligned}$$

The last inequality follows from the definition of λ_ϵ immediately.

Put

$$K(\omega) = \max\left\{K_0(\omega), \frac{\mathrm{diam}\overline{V}}{r_0}, \frac{\max\{||D_xT_\epsilon(\omega)|| : x \in \overline{V}\}}{r_0^\alpha}\right\}$$

and then the inequality (4.3) immediately follows.

Step 2: We claim that

$$\int \log \Lambda_n^\epsilon \mathrm{d}\mathbb{P} > 0$$

for some $n \geq 1$, where $\Lambda_n^\epsilon(\omega) = \min_{x \in J_\epsilon(\omega)} \prod_{k=0}^{n-1} \lambda_\epsilon(\vartheta^k\omega, T_\epsilon(k, \omega)x)$.

Recall that J is an average conformal repeller for f. It is easy to see that $\lim_{\epsilon \to 0} \log \Lambda_n^\epsilon = \min_{x \in J}\{\log m(D_{f^{n-1}x}f) + \cdots + \log m(D_xf)\} > 0$ in probability. By making $\mathcal{U}(f)$ smaller if necessary we have that $|\log(|D_xTe|)|$ is uniformly bounded for all $T \in$

$\mathcal{U}(f), x \in \overline{V}$, and $e \in T_xM$ with $|e| = 1$. Thus $\lim_{\epsilon \to 0} \log \Lambda_n^\epsilon = \min_{x \in J}\{\log m(D_{f^{n-1}x} f) + \cdots + \log m(D_x f)\} > 0$ also in $L^1(\mathbb{P})$, which implies $\sup_{n \geq 1} \frac{1}{n} \int \log \Lambda_n^\epsilon d\mathbb{P} > 0$ for sufficiently small ϵ. This finishes the proof of the claim.

Step 3: We claim that T_ϵ is strongly expansive. By remark 2 we know f is expanding on J, then there exists a neighborhood V of J, a constant $c > 0$, and an integer $n \geq 1$ such that $|D_x f^n(e)| \geq 1 + c$ for every $x \in V$ and $e \in T_xM$ with $|e| = 1$. We can choose $\mathcal{U}(f)$ in such a way that $|D_x(T_n \circ \cdots \circ T_1)(e)| \geq 1 + \frac{c}{2}$ whenever $T_1, \ldots, T_n \in \mathcal{U}(f), x \in V$, and $e \in T_xM$ with $|e| = 1$, and that $J_\epsilon(\omega) \subset V$ for every $\epsilon > 0$ and $\omega \in \Omega$. Then T_ϵ is uniformly expanding and thus strongly expanding on the bundle $\{J_\epsilon(\omega)\}_{\omega \in \Omega}$.

Step 4: Let $L \subset J$ be a compact subset with $fL = L$. We apply corollary 3.5 in [4] to the bundle RDS T_ϵ on $J_\epsilon = \{(\omega, x) : x \in h_\epsilon(\omega)L\}$. Let π_ϵ denote the pressure functional of T_ϵ restricted to J_ϵ, then we can get that there exist $s_1^\epsilon \geq t_1^\epsilon \geq 0$ such that

$$\pi_\epsilon(-t_1^\epsilon \log \eta_\epsilon) = 0 = \pi_\epsilon(-s_1^\epsilon \log \lambda_\epsilon)$$

and

$$t_1^\epsilon \leq \dim_H(h_\epsilon(\omega)L) \leq s_1^\epsilon \quad \mathbb{P}\text{-a.s.}$$

If we consider the system $T_\epsilon(n, \omega)$ and $\log \|D_x T_\epsilon(n, \omega)\|$, $\log m(D_x T_\epsilon(n, \omega))$ for every $n > 0$, and let $\pi_{n,\epsilon}$ denote the pressure functional of $T_\epsilon(n, \omega)$ restricted to J_ϵ, then we can get that

$$t_n^\epsilon \leq \dim_H(h_\epsilon(\omega)L) \leq s_n^\epsilon \quad \mathbb{P}\text{-a.s.}$$

where $t_n^\epsilon, s_n^\epsilon$ satisfying $\pi_{n,\epsilon}(-t_n^\epsilon \log \|D_x T_\epsilon(n, \omega)\|) = 0 = \pi_{n,\epsilon}(-s_n^\epsilon \log m(D_x T_\epsilon(n, \omega)))$. Furthermore, by theorem 2.2 the Hausdorff dimension of L is the unique $t_0 \geq 0$ with $\pi_{f|_L}(-t_0 \Psi) = 0$, where $\Psi = \{\log m(D_x f^n) : x \in L, n \in \mathbb{N}\}$.

Step 5: We first note that, for each fixed positive integer n, we have

$$\lim_{\epsilon \to 0} \| \log \|D_x T_\epsilon(n, \omega)\| - \log \|D_x f^n\| \|_\alpha = 0$$

and

$$\lim_{\epsilon \to 0} \| \log m(D_x T_\epsilon(n, \omega)) - \log m(D_x f^n)\|_\alpha = 0 \ in \ L^1(\mathbb{P}),$$

since $|\log \|D_x T_\epsilon(n, \omega)^\pm\||$ is uniformly bounded for the fixed positive integer n. By the proposition 3.1, for each fixed positive integer n, we have

$$\lim_{\epsilon \to 0} \frac{1}{n} \pi_{n,\epsilon}(-t \log \|D_x T_\epsilon(n, \omega)\|) = \frac{1}{n} \pi_{f^n|_L}(-t \log \|D_x f^n\|)$$

and

$$\lim_{\epsilon \to 0} \frac{1}{n} \pi_{n,\epsilon}(-t \log m(D_x T_\epsilon(n, \omega))) = \frac{1}{n} \pi_{f^n|_L}(-t \log m(D_x f^n))$$

for each $t \geq 0$. Moreover, by proposition 2.2 in [7] and proposition 2.3 we have

$$\lim_{n\to\infty} \frac{1}{n}\pi_{f^n|_L}(-t\log\|D_xf^n\|) = \lim_{n\to\infty} \frac{1}{n}\pi_{f^n|_L}(-t\log m(D_xf^n)) = \pi_{f|_L}(-t\Psi).$$

Hence, we obtain for each $t \geq 0$ that

$$\begin{aligned}\lim_{n\to\infty}\lim_{\epsilon\to 0}\frac{1}{n}\pi_{n,\epsilon}(-t\log\|D_xT_\epsilon(n,\omega)\|) &= \lim_{n\to\infty}\lim_{\epsilon\to 0}\frac{1}{n}\pi_{n,\epsilon}(-t\log m(D_xT_\epsilon(n,\omega))) \\ &= \pi_{f|_L}(-t\Psi).\end{aligned}$$

Step 6: To complete the proof, given $\delta > 0$. Since $t \mapsto \pi_{f|_L}(-t\Psi)$ is strictly decreasing, there exist $N > 0, \epsilon_0 > 0$ such that for each $\epsilon \leq \epsilon_0$, we have

$$\pi_{N,\epsilon}(-(t_0+\delta)\log\|D_xT_\epsilon(N,\omega)\|) < 0 < \pi_{N,\epsilon}(-(t_0-\delta)\log\|D_xT_\epsilon(N,\omega)\|)$$

and

$$\pi_{N,\epsilon}(-(t_0+\delta)\log m(D_xT_\epsilon(N,\omega))) < 0 < \pi_{N,\epsilon}(-(t_0-\delta)\log m(D_xT_\epsilon(N,\omega))).$$

This immediately implies

$$t_0 - \delta < t_N^\epsilon \leq \dim_H(h_\epsilon(\omega)L) \leq s_N^\epsilon < t_0 + \delta. \tag{4.4}$$

The desired result then immediately follows. ☐

Remark 4. *(1) In [4], Bogenschütz and Ochs proved that the Hausdorff dimension of a conformal repeller is stable under random perturbations. Using their ideas, we show that the same is true for average conformal repeller. The differences between theorem 4.1 and Bogenschütz and Ochs's theorem are:*

i) In order to use the corollary 3.5 in [4], it is the same from step 1 to step 3;

ii) In order to prove the Hausdorff dimension of average conformal repeller is stable under random perturbation, we should consider the iteration of the RDS from step 4 to step 6. And this process need the technic of sub-additive topological pressure and sup-additive topological pressure. In [4], the authors need not consider the iteration of the RDS, so they need only additive topological pressure.

(2) Since the bundle T_ϵ is uniformly expanding on the bundle $\{J_\epsilon(\omega)\}_{\omega\in\Omega}$, the result in [15] told us that there exists a equilibrium states of the topological pressure π_ϵ. Then modifying subtly the proof in [1] we can get that the zero of the sub-additive topological pressure is the upper bound of the Hausdorff dimension of the bundle $\{J_\epsilon(\omega)\}_{\omega\in\Omega}$.

Proposition 4.2. *Under the conditions of theorem 4.1, we have*

$$\lim_{\epsilon\to 0} h_{top}^{(r)}(T_\epsilon) = h_{top}(f),$$

where $h_{top}^{(r)}(T_\epsilon)$ denote the topological entropy of the random dynamical system T_ϵ generated by the random perturbation of f and $h_{top}(f)$ denote the classical topological entropy of deterministic dynamical system.

Proof. This can be immediately deduced from proposition 3.1 by taking the potential functions to be the zero-valued functions. □

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